

On the limitation of spectral methods: From the Gaussian hidden clique problem to rank one perturbations of Gaussian tensors

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Abstract

We consider the following detection problem: given a realization of a symmetric matrix \mathbf{X} of dimension n , distinguish between the hypothesis that all upper triangular variables are i.i.d. Gaussians variables with mean 0 and variance 1 and the hypothesis where \mathbf{X} is the sum of such matrix and an independent rank-one perturbation.

This setup applies to the situation where under the alternative, there is a planted principal submatrix \mathbf{B} of size L for which all upper triangular variables are i.i.d. Gaussians with mean 1 and variance 1, whereas all other upper triangular elements of \mathbf{X} not in \mathbf{B} are i.i.d. Gaussians variables with mean 0 and variance 1. We refer to this as the ‘Gaussian hidden clique problem.’

When $L = (1 + \epsilon)\sqrt{n}$ ($\epsilon > 0$), it is possible to solve this detection problem with probability $1 - o_n(1)$ by computing the spectrum of \mathbf{X} and considering the largest eigenvalue of \mathbf{X} . We prove that this condition is tight in the following sense: when $L < (1 - \epsilon)\sqrt{n}$ no algorithm that examines only the eigenvalues of \mathbf{X} can detect the existence of a hidden Gaussian clique, with error probability vanishing as $n \rightarrow \infty$.

We prove this result as an immediate consequence of a more general result on rank-one perturbations of k -dimensional Gaussian tensors. In this context we establish a lower bound on the critical signal-to-noise ratio below which a rank-one signal cannot be detected.

1 Introduction

Consider the following detection problem. One is given a symmetric matrix $\mathbf{X} = \mathbf{X}(n)$ of dimension n , such that the $\binom{n}{2} + n$ entries $(\mathbf{X}_{i,j})_{i \leq j}$ are *mutually independent* random variables. Given (a realization of) \mathbf{X} one would like to distinguish between the hypothesis that all random variables $\mathbf{X}_{i,j}$ have the same distribution F_0 to the hypothesis where there is a set $U \subseteq [n]$ so that all random variables in the submatrix $\mathbf{X}_U := (\mathbf{X}_{s,t} : s, t \in U)$ have a distribution F_1 which is different from the distribution of all other elements in \mathbf{X} which are still distributed as F_0 .

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The same problem was recently studied in [1, 11] and, for the of the asymmetric case (where no symmetry assumption is imposed on the independent entries of \mathbf{X}), in [7, 26, 28]. We refer to Section 6 for further discussion of the related literature. An intriguing outcome of these works is that, while the two hypothesis are statistically distinguishable as soon as $L \geq C \log n$ (for C a sufficiently large constant) [8], practical algorithms require significantly larger L . This motivates the study of restricted classes of tests. In this paper we study the class of spectral (or eigenvalue-based) tests detecting the signal. Our proof technique naturally allow to consider two further generalizations of this problem that are of independent interests. We briefly summarize our results below.

The Gaussian hidden clique problem. This is a special case of the above hypothesis testing setting, whereby $F_0 = \mathcal{N}(0, 1)$ and $F_1 = \mathcal{N}(1, 1)$ (entries on the diagonal are defined slightly differently for simplifying calculations). Here and below $\mathcal{N}(m, \sigma^2)$ denote the Gaussian distribution of mean m and variance σ^2 . Equivalently, let \mathbf{Z} be a random matrix from the Gaussian Orthogonal Ensemble (GOE) i.e. $\mathbf{Z}_{ij} \sim \mathcal{N}(0, 1/n)$ independently for $i < j$, and $\mathbf{Z}_{ii} \sim \mathcal{N}(0, 2/n)$. Then, under hypothesis $H_{1,L}$ we have $\mathbf{X} = n^{-1/2} \mathbf{1}_U \mathbf{1}_U^\top + \mathbf{Z}$ ($\mathbf{1}_U$ being the indicator vector on U , and $|U| = L$), and under hypothesis H_0 , $\mathbf{X} = \mathbf{Z}$ (the factor n in the normalization is for technical convenience).

We then consider the following restricted hypothesis testing question. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the ordered eigenvalues of \mathbf{X} . *Is there a test that depends only on $\lambda_1, \dots, \lambda_n$ and that distinguishes H_0 from $H_{1,L}$ ‘reliably,’ i.e. with error probability converging to 0 as $n \rightarrow \infty$?* Notice that the eigenvalues distribution does not depend on U as long as this is independent from the noise \mathbf{Z} . We can therefore think of U as fixed for this question.

If $L \geq (1 + \varepsilon)\sqrt{n}$ then [17] implies that a simple test checking whether $\lambda_1 \geq 2 + \delta$ for some $\delta = \delta(\varepsilon) > 0$ is reliable. We prove that this result is tight, in the sense that no spectral test is reliable for $L \leq (1 - \varepsilon)\sqrt{n}$.

Rank-one matrices in Gaussian noise. Our proof technique builds on a simple remark. Since the noise \mathbf{Z} is invariant under orthogonal transformations¹, the above question is equivalent to the following testing problem. For $\beta \in \mathbb{R}_{\geq 0}$, and $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v}\|_2 = 1$ a uniformly random unit vector, test $H_0: \mathbf{X} = \mathbf{Z}$ versus $H_1, \mathbf{X} = \beta \mathbf{v} \mathbf{v}^\top + \mathbf{Z}$. (The correspondence between the two problems yields $\beta = L/\sqrt{n}$.)

Again, this problem (and a closely related asymmetric version [32]) has been studied in the literature, and it follows from [17] that a reliable test exists for $\beta \geq 1 + \varepsilon$. We provide a simple proof (based on the second moment method) that no test is reliable for $\beta < 1 + \varepsilon$.

Rank-one tensors in Gaussian noise. It turns out that the same proof applies to an even more general problem: detecting a rank-one signal in a noisy tensor. We carry out our analysis in this more general setting for two reasons. First, we think that this clarifies the what aspects of the model are important for our proof technique to apply. Second, the problem estimating tensors from noisy data has attracted significant interest recently within the machine learning community [22, 31].

More precisely, we consider a noisy tensor $\mathbf{X} \in \bigotimes^k \mathbb{R}^n$, of the form $\mathbf{X} = \beta \mathbf{v}^{\otimes k} + \mathbf{Z}$, where \mathbf{Z} is Gaussian noise, and \mathbf{v} is a random unit vector. We consider the problem of testing this hypothesis against $H_0: \mathbf{X} = \mathbf{Z}$. We establish a threshold $\beta_k^{2\text{nd}}$ such that no test can be reliable for $\beta < \beta_k^{2\text{nd}}$ (in particular $\beta_2^{2\text{nd}} = 1$). We establish an analogous result for the asymmetric case as well. Two differences are worth remarking for $k \geq 3$ with respect to the more familiar matrix case $k = 2$.

¹By this we mean that, for any orthogonal matrix $\mathbf{R} \in O(n)$, independent of \mathbf{Z} , $\mathbf{R} \mathbf{Z} \mathbf{R}^\top$ is distributed as \mathbf{Z} .

First, we do not expect the second moment bound $\beta_k^{2\text{nd}}$ to be tight, i.e. a reliable test to exist for all $\beta > \beta_k^{2\text{nd}}$. On the other hand, we can show that it is tight up to a universal (k and n independent) constant. Second, below $\beta_k^{2\text{nd}}$ the problem is more difficult than the matrix version below $\beta_2^{2\text{nd}} = 1$: not only no reliable test exists but, asymptotically, any test behaves asymptotically as random guessing.

2 Main result for spectral detection

Let \mathbf{Z} be a GOE matrix as defined in the previous section. Equivalently if \mathbf{G} is an (asymmetric) matrix with i.i.d. entries $\mathbf{G}_{i,j} \sim \mathcal{N}(0, 1)$,

$$\mathbf{Z} = \frac{1}{\sqrt{2n}}(\mathbf{G} + \mathbf{G}^\top). \quad (1)$$

For a deterministic sequence of vectors $\mathbf{v}(n)$, $\|\mathbf{v}(n)\|_2 = 1$, we consider the two hypotheses

$$\begin{cases} H_0 : & \mathbf{X} = \mathbf{Z}, \\ H_{1,\beta} : & \mathbf{X} = \beta \mathbf{v} \mathbf{v}^\top + \mathbf{Z}. \end{cases} \quad (2)$$

A special example is provided by the *Gaussian hidden clique* problem in which case $\beta = L/\sqrt{n}$ and $\mathbf{v} = \mathbf{1}_U/\sqrt{L}$ for some set $U \subseteq [n]$, $|U| = L$,

$$\begin{cases} H_0 : & \mathbf{X} = \mathbf{Z}, \\ H_{1,L} : & \mathbf{X} = \frac{1}{\sqrt{n}} \mathbf{1}_U \mathbf{1}_U^\top + \mathbf{Z}. \end{cases} \quad (3)$$

Observe that the distribution of eigenvalues of \mathbf{X} , under either alternative, is invariant to the choice of the vector \mathbf{v} (or subset U), as long as the norm of \mathbf{v} is kept fixed. Therefore, any successful spectral algorithm will distinguish between H_0 and $H_{1,\beta}$ but not give any information on the vector \mathbf{v} (or subset U , in the case of $H_{1,L}$).

We let $Q_0 = Q_0(n)$ (respectively, $Q_1 = Q_1(n)$) denote the distribution of the eigenvalues of \mathbf{X} under H_0 (respectively $H_1 = H_{1,\beta}$ or $H_{1,L}$).

A *spectral statistical test* for distinguishing between H_0 and H_1 (or simply a spectral test) is a measurable map $T_n : (\lambda_1, \dots, \lambda_n) \mapsto \{0, 1\}$. To formulate precisely what we mean by the word *distinguish*, we introduce the following notion.

Definition 1. For each $n \in \mathbb{N}$, let $\mathbb{P}_{0,n}, \mathbb{P}_{1,n}$ be two probability measures on the same measure space $(\Omega_n, \mathcal{F}_n)$. We say that the sequence $(\mathbb{P}_{1,n})$ is *contiguous with respect to* $(\mathbb{P}_{0,n})$ if, for any sequence of events $A_n \in \mathcal{F}_n$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_{0,n}(A_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}_{1,n}(A_n) = 0. \quad (4)$$

Note that contiguity is not in general a symmetric relation.

In the context of the spectral statistical tests described above, the sequences A_n in Definition 1 (with $P_n = Q_0(n)$ and $Q_n = Q_1(n)$) can be put in correspondence with spectral statistical tests T_n by taking $A_n = \{(\lambda_1, \dots, \lambda_n) : T_n(\lambda_1, \dots, \lambda_n) = 0\}$. We will thus say that H_1 is *spectrally contiguous* with respect to H_0 if Q_n is contiguous with respect to P_n .

Our main result on the Gaussian hidden clique problem is the following.

Theorem 1. *For any sequence $L = L(n)$ satisfying $\limsup_{n \rightarrow \infty} L(n)/\sqrt{n} < 1$, the hypotheses $H_{1,L}$ are spectrally contiguous with respect to H_0 .*

Equivalently for any sequence of vectors $\mathbf{v}(n)$, and any $\beta = \beta(n)$ satisfying $\limsup_{n \rightarrow \infty} \beta(n) < 1$, the hypotheses $H_{1,\beta}$ are spectrally contiguous with respect to H_0 .

Several remarks are in order with respect to Definition 1 and Theorem 1. First, we rule out arbitrary spectral tests - not just tests that have polynomial running time (in n). Second, we *do not* rule out the existence of a spectral test (or even a spectral test running in polynomial time) that distinguishes between H_0 and $H_{1,L}$ with total probability of error $\mathbb{P}_0(T = 1) + \mathbb{P}_1(T = 0) \leq 1 - \delta$. Indeed, as discussed in Section 4.1, it is quite easy to construct such a test.

Finally Theorem 1 provides an example of two family of distributions H_0 and $H_{1,L}$ such that the total variation distance between H_0 and $H_{1,L}$ is $1 - o_n(1)$ whereas H_0 and $H_{1,L}$ are spectrally contiguous.

2.1 Contiguity and integrability

Contiguity is related to a notion of uniform absolute continuity of measures. Recall that a probability measure μ on a measure space is *absolutely continuous* with respect to another probability measure ν if for every measurable set A , $\nu(A) = 0$ implies that $\mu(A) = 0$, in which case there exists a ν -integrable, non-negative function $f \equiv \frac{d\mu}{d\nu}$ (the *Radon-Nikodym derivative* of μ with respect to ν), so that $\mu(A) = \int_A f d\nu$ for every measurable set A . We then have the following known useful fact, which will be the basis for proving contiguity, and whose proof is given for completeness.

Lemma 2. *Within the setting of Definition 1, assume that $\mathbb{P}_{1,n}$ is absolutely continuous with respect to $\mathbb{P}_{0,n}$, and denote by $\Lambda_n \equiv \frac{d\mathbb{P}_{1,n}}{d\mathbb{P}_{0,n}}$ its Radon-Nikodym derivative.*

- (a) *If $\limsup_{n \rightarrow \infty} \mathbb{E}_{0,n}(\Lambda_n^2) < \infty$, then $(\mathbb{P}_{1,n})$ is contiguous with respect to $(\mathbb{P}_{0,n})$.*
- (b) *If $\lim_{n \rightarrow \infty} \mathbb{E}_{0,n}(\Lambda_n^2) = 1$, then $\lim_{n \rightarrow \infty} \|\mathbb{P}_{0,n} - \mathbb{P}_{1,n}\|_{\text{TV}} = 0$, where $\|\cdot\|_{\text{TV}}$ denotes the total variation distance, i.e.*

$$\|\mathbb{P}_{0,n} - \mathbb{P}_{1,n}\|_{\text{TV}} \equiv \sup_A |\mathbb{P}_{0,n}(A) - \mathbb{P}_{1,n}(A)|.$$

Proof. (a) Let A_n be any sequence of events such that $\mathbb{P}_{0,n}(A_n) \rightarrow 0$. Then

$$\mathbb{P}_{1,n}(A_n) = \mathbb{E}_{0,n}\{\Lambda_n \mathbf{1}_{A_n}\} \leq \mathbb{E}_{0,n}\{\Lambda_n^2\}^{1/2} \mathbb{P}_{0,n}(A_n)^{1/2} \rightarrow 0, \quad (5)$$

where the last limit follows since $\mathbb{E}_{0,n}\{\Lambda_n^2\} \leq C$ for all n large enough.

(b) Note that $\mathbb{E}_{0,n}\Lambda_n = 1$, whence

$$2\|\mathbb{P}_{0,n} - \mathbb{P}_{1,n}\|_{\text{TV}} = \mathbb{E}_{0,n}\{|\Lambda_n - 1|\} \leq \mathbb{E}_{0,n}\{(\Lambda_n - 1)^2\}^{1/2} = \sqrt{\mathbb{E}_{0,n}(\Lambda_n^2) - 1}. \quad (6)$$

□

2.2 Method and structure of the paper

Consider problem (2). We use the fact that the law of the eigenvalues under both H_0 and $H_{1,\beta}$ are invariant under conjugations by a orthogonal matrix. Once we conjugate matrices sampled

under the hypothesis $H_{1,\beta}$ by an independent orthogonal matrix sampled according to the Haar distribution, we get a matrix distributed as

$$\mathbf{X} = \beta \mathbf{v} \mathbf{v}^\top + \mathbf{Z}, \quad (7)$$

where \mathbf{u} is uniform on the n -dimensional sphere, and \mathbf{Z} is a GOE matrix (with off-diagonal entries of variance $1/n$). Letting $\mathbb{P}_{1,n}$ denote the law of $\beta \mathbf{u} \mathbf{u}^\top + \mathbf{Z}$ and $\mathbb{P}_{0,n}$ denote the law of \mathbf{Z} , we show that $\mathbb{P}_{1,n}$ is contiguous with respect to $\mathbb{P}_{0,n}$, which implies that the law of eigenvalues $Q_1(n)$ is contiguous with respect to $Q_0(n)$.

To show the contiguity, we consider a more general setup, of independent interest, of Gaussian tensors of order k , and in that setup show that the Radon-Nikodym derivative $\Lambda_{n,L} = \frac{d\mathbb{P}_{1,n}}{d\mathbb{P}_{0,n}}$ is uniformly square integrable under $\mathbb{P}_{0,n}$; an application of Lemma 2 then quickly yields Theorem 1.

The structure of the paper is as follows. In the next section, we define formally the detection problem for a symmetric tensor of order $k \geq 2$. We show the existence of a threshold under which detection is not possible (Theorem 3), and show how Theorem 1 follows from this. Section 4 is devoted to the proof of Theorem 3, and concludes with some additional remarks and consequences of Theorem 3. Section 5 treats the case of asymmetric Gaussian tensors. Finally, Section 6 is devoted to a description of the relation between the Gaussian hidden clique problem and hidden clique problem in computer science, and related literature.

3 A symmetric tensor model and a reduction

Exploiting rotational invariance, we will reduce the spectral detection problem to a detection problem involving a standard detection problem between random matrices. Since the latter generalizes to a tensor setup, we first introduce a general Gaussian hypothesis testing for k -tensors, which is of independent interest. We then explain how the spectral detection problem reduces to the special case of $k = 2$.

3.1 Preliminaries and notation

We use lower-case boldface for vectors (e.g. \mathbf{u} , \mathbf{v} , and so on) and upper-case boldface for matrices and tensors (e.g. \mathbf{X} , \mathbf{Z} , and so on). The ordinary scalar product and ℓ_p norm over vectors are denoted by $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n \mathbf{u}_i \mathbf{v}_i$, and $\|\mathbf{v}\|_p$. We write \mathbb{S}^{n-1} for the unit sphere in n dimensions

$$\mathbb{S}^{n-1} \equiv \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}. \quad (8)$$

Given $\mathbf{X} \in \bigotimes^k \mathbb{R}^n$ a real k -th order tensor, we let $\{\mathbf{X}_{i_1, \dots, i_k}\}_{i_1, \dots, i_k}$ denote its coordinates. The outer product of two tensors is $\mathbf{X} \otimes \mathbf{Y}$, and, for $\mathbf{v} \in \mathbb{R}^n$, we define $\mathbf{v}^{\otimes k} = \mathbf{v} \otimes \dots \otimes \mathbf{v} \in \bigotimes^k \mathbb{R}^n$ as the k -th outer power of \mathbf{v} . We define the inner product of two tensors $\mathbf{X}, \mathbf{Y} \in \bigotimes^k \mathbb{R}^n$ as

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i_1, \dots, i_k \in [n]} \mathbf{X}_{i_1, \dots, i_k} \mathbf{Y}_{i_1, \dots, i_k}. \quad (9)$$

We define the Frobenius (Euclidean) norm of a tensor \mathbf{X} by $\|\mathbf{X}\|_F = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle}$, and its operator norm by

$$\|\mathbf{X}\|_{op} \equiv \max\{\langle \mathbf{X}, \mathbf{u}_1 \otimes \dots \otimes \mathbf{u}_k \rangle : \forall i \in [k], \|\mathbf{u}_i\|_2 \leq 1\}. \quad (10)$$

It is easy to check that this is indeed a norm. For the special case $k = 2$, it reduces to the ordinary ℓ_2 matrix operator norm (equivalently, to the largest singular value of \mathbf{X}).

For a permutation $\pi \in \mathfrak{S}_k$, we will denote by \mathbf{X}^π the tensor with permuted indices $\mathbf{X}_{i_1, \dots, i_k}^\pi = \mathbf{X}_{\pi(i_1), \dots, \pi(i_k)}$. We call the tensor \mathbf{X} *symmetric* if, for any permutation $\pi \in \mathfrak{S}_k$, $\mathbf{X}^\pi = \mathbf{X}$. It is proved [33] that, for symmetric tensors, for symmetric tensors we have the equivalent representation

$$\|\mathbf{X}\|_{op} \equiv \max\{|\langle \mathbf{X}, \mathbf{u}^{\otimes k} \rangle| : \|\mathbf{u}\|_2 \leq 1\}. \quad (11)$$

We define $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \infty$ with the usual conventions of arithmetic operations.

3.2 The symmetric tensor model and main result

We denote by $\mathbf{G} \in \bigotimes^k \mathbb{R}^n$ a tensor with independent and identically distributed entries $\mathbf{G}_{i_1, \dots, i_k} \sim \mathcal{N}(0, 1)$ (note that this tensor is not symmetric).

We define the *symmetric standard normal* noise tensor $\mathbf{Z} \in \bigotimes^k \mathbb{R}^n$ by

$$\mathbf{Z} = \frac{1}{k!} \sqrt{\frac{2}{n}} \sum_{\pi \in \mathfrak{S}_k} \mathbf{G}^\pi. \quad (12)$$

Note that the subset of entries with unequal indices form an i.i.d. collection $\{\mathbf{Z}_{i_1, i_2, \dots, i_k}\}_{i_1 < \dots < i_k} \sim \mathcal{N}(0, 2/(n(k!)))$.

With this normalization, we have, for any symmetric tensor $\mathbf{A} \in \bigotimes^k \mathbb{R}^n$

$$\mathbb{E}\{e^{\langle \mathbf{A}, \mathbf{Z} \rangle}\} = \exp\left\{\frac{1}{n} \|\mathbf{A}\|_F^2\right\}. \quad (13)$$

We will also use the fact that \mathbf{Z} is invariant in distribution under conjugation by orthogonal transformations, that is, that for any orthogonal matrix $U \in O(n)$, $\{\mathbf{Z}_{i_1, \dots, i_k}\}$ has the same distribution as $\{\sum_{j_1, \dots, j_k} \left(\prod_{\ell=1}^k U_{i_\ell, j_\ell}\right) \cdot \mathbf{Z}_{j_1, \dots, j_k}\}$.

Given a parameter $\beta \in \mathbb{R}_{\geq 0}$, we consider the following model for a random symmetric tensor \mathbf{X} :

$$\mathbf{X} \equiv \beta \mathbf{v}^{\otimes k} + \mathbf{Z}, \quad (14)$$

with \mathbf{Z} a standard normal tensor, and \mathbf{v} uniformly distributed over the unit sphere \mathbb{S}^{n-1} . In the case $k = 2$ this is the standard rank-one deformation of a GOE matrix.

We let $\mathbb{P}_\beta = \mathbb{P}_\beta^{(k)}$ denote the law of \mathbf{X} under model (14).

Theorem 3. *For $k \geq 2$, let*

$$\beta_k^{2\text{nd}} \equiv \inf_{q \in (0, 1)} \sqrt{-\frac{1}{q^k} \log(1 - q^2)}. \quad (15)$$

Assume $\beta < \beta_k^{2\text{nd}}$. Then, for any $k \geq 3$, we have

$$\lim_{n \rightarrow \infty} \|\mathbb{P}_\beta - \mathbb{P}_0\|_{\text{TV}} = 0. \quad (16)$$

Further, for $k = 2$ and $\beta < \beta_k^{2\text{nd}} = 1$, \mathbb{P}_β is contiguous with respect to \mathbb{P}_0 .

The notation $\beta_k^{2\text{nd}}$ refers to the fact that this is the threshold for the second moment method to work.

3.3 Reduction of spectral detection to the symmetric tensor model, $k = 2$, and proof of Theorem 1

Recall that in the setup of Theorem 1, $Q_{0,n}$ is the law of the eigenvalues of \mathbf{X} under H_0 and $Q_{1,n}$ is the law of the eigenvalues of \mathbf{X} under $H_{1,L}$. Then $Q_{1,n}$ is invariant by conjugation of orthogonal matrices. Therefore, the detection problem is not changed if we replace $\mathbf{X} = n^{-1/2} \mathbf{1}_U \mathbf{1}_U^\top + \mathbf{Z}$ by

$$\hat{\mathbf{X}} \equiv \mathbf{R} \mathbf{X} \mathbf{R}^\top = \frac{1}{\sqrt{n}} \mathbf{R} \mathbf{1}_U (\mathbf{R} \mathbf{1}_U)^\top + \mathbf{R} \mathbf{Z} \mathbf{R}^\top, \quad (17)$$

where $\mathbf{R} \in O(n)$ is an orthogonal matrix sampled according to the Haar measure. A direct calculations yields

$$\hat{\mathbf{X}} = \beta \mathbf{u} \mathbf{u}^\top + \tilde{\mathbf{Z}}, \quad (18)$$

where \mathbf{u} is uniform on the n dimensional sphere, $\beta = L/\sqrt{n}$, and $\tilde{\mathbf{Z}}$ is a GOE matrix (with off-diagonal entries of variance $1/n$). Furthermore, \mathbf{u} and $\tilde{\mathbf{Z}}$ are independent of one another.

Let $\mathbb{P}_{1,n}$ be the law of $\hat{\mathbf{X}}$. Note that $\mathbb{P}_{1,n} = \mathbb{P}_\beta^{(k=2)}$ with $\beta = L/\sqrt{n}$. We can relate the detection problem of H_0 vs. $H_{1,L}$ to the detection problem of $\mathbb{P}_{0,n}$ vs. $\mathbb{P}_{1,n}$ as follows.

Lemma 4. (a) If $\mathbb{P}_{1,n}$ is contiguous with respect to $\mathbb{P}_{0,n}$ then $H_{1,L}$ is spectrally contiguous with respect to H_0 .

(b) We have

$$\|Q_{0,n} - Q_{1,n}\|_{\text{TV}} \leq \|\mathbb{P}_{0,n} - \mathbb{P}_{1,n}\|_{\text{TV}}.$$

Proof. (a) Let \mathcal{F} be the σ -algebra generated by all rotation-invariant events, and let \mathcal{G} be the σ -algebra generated by the ordered eigenvalues. Then $\mathcal{F} = \mathcal{G}$.

Let $A_n \in \mathcal{G}$ be (spectral) events so that $Q_{0,n}(A_n) \rightarrow_{n \rightarrow \infty} 0$. Note that (with some abuse of notation) we have that $Q_{0,n}$ coincides with $\mathbb{P}_{0,n}$ and $Q_{1,n}$ coincides with $\mathbb{P}_{1,n}$ on \mathcal{G} . Then $\mathbb{P}_{1,n}(A_n) \rightarrow_{n \rightarrow \infty} 0$ by assumption and therefore $Q_{1,n}(A_n) = \mathbb{P}_{1,n}(A_n) \rightarrow_{n \rightarrow \infty} 0$, proving the contiguity of $Q_{1,n}$ with respect to $Q_{0,n}$.

(b) We have

$$\begin{aligned} \sup_A |\mathbb{P}_{0,n}(A) - \mathbb{P}_{1,n}(A)| &\geq \sup_{A \in \mathcal{F}} |\mathbb{P}_{0,n}(A) - \mathbb{P}_{1,n}(A)| = \sup_{A \in \mathcal{G}} |\mathbb{P}_{0,n}(A) - \mathbb{P}_{1,n}(A)| \\ &= \sup_{A \in \mathcal{B}} |Q_{1,n}(A) - Q_{0,n}(A)|, \end{aligned} \quad (19)$$

where \mathcal{B} is the Borel σ -algebra in $\Sigma^n := \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$. \square

Remark: The first inequality in (19) (and hence the inequality in point (b) of the statement) is actually an equality, but we will not use this fact.

In view of Lemma 4, Theorem 1 is an immediate consequence of Theorem 3.

4 Proof of Theorem 3

The proof uses the following large deviations lemma, which follows, for instance, from [13, Proposition 2.3].

Lemma 5. Let \mathbf{v} a uniformly random vector on the unit sphere \mathbb{S}^{n-1} and let $\langle \mathbf{v}, \mathbf{e}_1 \rangle$ be its first coordinate. Then, for any interval $[a, b]$ with $-1 \leq a < b \leq 1$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\langle \mathbf{v}, \mathbf{e}_1 \rangle \in [a, b]) = \max \left\{ \frac{1}{2} \log(1 - q^2) : q \in [a, b] \right\}. \quad (20)$$

Proof of Theorem 3. We denote by Λ the Radon-Nikodym derivative of \mathbb{P}_β with respect to \mathbb{P}_0 . By definition $\mathbb{E}_0 \Lambda = 1$. It is easy to derive the following formula

$$\Lambda = \int \exp \left\{ -\frac{n\beta^2}{4} + \frac{n\beta}{2} \langle \mathbf{X}, \mathbf{v}^{\otimes k} \rangle \right\} \mu_n(d\mathbf{v}). \quad (21)$$

where μ_n is the uniform measure on \mathbb{S}^{n-1} . Squaring and using (13), we get

$$\begin{aligned} \mathbb{E}_0 \Lambda^2 &= e^{-n\beta^2/2} \int \mathbb{E}_0 \exp \left\{ \frac{n\beta}{2} \langle \mathbf{X}, \mathbf{v}_1^{\otimes k} + \mathbf{v}_2^{\otimes k} \rangle \right\} \mu_n(d\mathbf{v}_1) \mu_n(d\mathbf{v}_2) \\ &= e^{-n\beta^2/2} \int \exp \left\{ \frac{n\beta^2}{4} \|\mathbf{v}_1^{\otimes k} + \mathbf{v}_2^{\otimes k}\|_F^2 \right\} \mu_n(d\mathbf{v}_1) \mu_n(d\mathbf{v}_2) \\ &= \int \exp \left\{ \frac{n\beta^2}{2} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle^k \right\} \mu_n(d\mathbf{v}_1) \mu_n(d\mathbf{v}_2) \\ &= \int \exp \left\{ \frac{n\beta^2}{2} \langle \mathbf{v}, \mathbf{e}_1 \rangle^k \right\} \mu_n(d\mathbf{v}), \end{aligned} \quad (22)$$

where in the first step we used (13) and in the last step, we used rotational invariance.

Let $F_\beta : [-1, 1] \rightarrow \mathbb{R}$ be defined by

$$F_\beta(q) \equiv \frac{\beta^2 q^k}{2} + \frac{1}{2} \log(1 - q^2). \quad (23)$$

Using Lemma 5, for any $-1 \leq a < b \leq 1$,

$$\int \exp \left\{ \frac{n\beta^2}{2} \langle \mathbf{v}, \mathbf{e}_1 \rangle^k \right\} \mathbb{I}(\langle \mathbf{v}, \mathbf{e}_1 \rangle \in [a, b]) \mu_n(d\mathbf{v}) = \exp \left\{ n \max_{q \in [a, b]} F_\beta(q) + o(n) \right\}. \quad (24)$$

It follows from the definition of $\beta_k^{2\text{nd}}$ that $\max_{|q| \geq \varepsilon} F_\beta(q) < 0$ for any $\varepsilon > 0$. Hence

$$\mathbb{E}_0 \Lambda^2 \leq \int \exp \left\{ \frac{n\beta^2}{2} \langle \mathbf{v}, \mathbf{e}_1 \rangle^k \right\} \mathbb{I}(|\langle \mathbf{v}, \mathbf{e}_1 \rangle| \leq \varepsilon) \mu_n(d\mathbf{v}) + e^{-c(\varepsilon)n}, \quad (25)$$

for some $c(\varepsilon) > 0$ and all n large enough. Next notice that, under μ_n , $\langle \mathbf{v}, \mathbf{e}_1 \rangle \stackrel{d}{=} G/(G^2 + Z_{n-1})^{1/2}$ where $G \sim \mathcal{N}(0, 1)$ and Z_{n-1} is a χ^2 with $n-1$ degrees of freedom independent of G . Then, letting $Z_n \equiv G^2 + Z_{n-1}$ (a χ^2 with n degrees of freedom)

$$\begin{aligned} \mathbb{E}_0 \Lambda^2 &\leq \mathbb{E} \left\{ \exp \left(\frac{n\beta^2}{2} \frac{|G|^k}{Z_n^{k/2}} \right) \mathbb{I}(|G/Z_n^{1/2}| \leq \varepsilon) \right\} + e^{-c(\varepsilon)n} \\ &\leq \mathbb{E} \left\{ \exp \left(\frac{n\beta^2}{2} \frac{|G|^k}{Z_n^{k/2}} \right) \mathbb{I}(|G/Z_n^{1/2}| \leq \varepsilon) \mathbb{I}(Z_{n-1} \geq n(1 - \delta)) \right\} \\ &\quad + e^{n\beta^2\varepsilon^k/2} \mathbb{P}\{Z_{n-1} \leq n(1 - \delta)\} + e^{-c(\varepsilon)n} \\ &\leq \mathbb{E} \left\{ \exp \left(\frac{n^{1-(k/2)}\beta^2}{2(1 - \delta)^{k/2}} |G|^k \right) \mathbb{I}(|G|^2 \leq 2\varepsilon n) \right\} + e^{n\beta^2\varepsilon^k/2} \mathbb{P}\{Z_{n-1} \leq n(1 - \delta)\} + e^{-c(\varepsilon)n} \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{2\varepsilon n} e^{C(\beta, \delta)n^{1-k/2}x^k - x^2/2} dx + e^{n\beta^2\varepsilon^k/2} \mathbb{P}\{Z_{n-1} \leq n(1 - \delta)\} + e^{-c(\varepsilon)n}, \end{aligned} \quad (26)$$

where $C(\beta, \delta) = \beta^2/(2(1 - \delta)^{k/2})$. Now, for any $\delta > 0$, we can (and will) choose ε small enough so that both $e^{n\beta^2\varepsilon^k/2}\mathbb{P}\{Z_{n-1} \leq n(1 - \delta)\} \rightarrow 0$ exponentially fast (by tail bounds on χ^2 random variables) and, if $k \geq 3$, the argument of the exponent in the integral in the right hand side of (26) is bounded above by $-x^2/4$, which is possible since the argument vanishes at $x^* = 2C(\beta, \delta)n^{1/2}$. Hence, for any $\delta > 0$, and all n large enough, we have

$$\mathbb{E}_0\Lambda^2 \leq \frac{2}{\sqrt{2\pi}} \int_0^{2\varepsilon n} e^{C(\beta, \delta)n^{1-k/2}x^k - x^2/2} dx + e^{-c(\delta)n}, \quad (27)$$

for some $c(\delta) > 0$.

Now, for $k \geq 3$ the integrand in (27) is dominated by $e^{-x^2/4}$ and converges pointwise (as $n \rightarrow \infty$) to 1. Therefore, since $\mathbb{E}_0\Lambda^2 \geq (\mathbb{E}_0\Lambda)^2 = 1$,

$$k \geq 3 : \quad \lim_{n \rightarrow \infty} \mathbb{E}_0\Lambda^2 = 1. \quad (28)$$

For $k = 2$, the argument is independent of n and can be integrated immediately, yielding (after taking the limit $\delta \rightarrow 0$)

$$k = 2 : \quad \limsup_{n \rightarrow \infty} \mathbb{E}_0\Lambda^2 \leq \frac{1}{\sqrt{1 - \beta^2}}. \quad (29)$$

(Indeed, the above calculation implies that the limit exists and is given by the right-hand side.)

The proof is completed by invoking Lemma 2. \square

4.1 Some remarks and consequences

Threshold values. In the table below we report the numerical values of $\beta_k^{2\text{nd}}$ for a few values of k . The exact value $\beta_k^{2\text{nd}} = 1$ for the matrix case $k = 2$ follows from $\log(1 - q^2) \leq -q^2$.

k	$\beta_k^{2\text{nd}}$
2	1
3	1.398841
4	1.566974
5	1.67676
6	1.757589
10	1.955118
100	2.595874

Also, it is not difficult to derive the asymptotics $\beta_k^{2\text{nd}} = \sqrt{\log(k/2)} + o_k(1)$ for large k .

Tightness of the threshold values. As mentioned in the introduction, for $k = 2$ and $\beta > 1$, it is known that the largest eigenvalue of \mathbf{X} , $\lambda_1(\mathbf{X})$ converges almost surely to $(\beta + 1/\beta)$ [17]. As a consequence $\|\mathbb{P}_0 - \mathbb{P}_\beta\|_{\text{TV}} \rightarrow 1$ for all $\beta > 1$: the second moment bound is tight.

For $k \geq 3$, it follows by the triangular inequality that $\|\mathbf{X}\|_{\text{op}} \geq \beta - \|\mathbf{Z}\|_{\text{op}}$, and further $\limsup_{n \rightarrow \infty} \|\mathbf{Z}\|_{\text{op}} \leq \mu_k$ almost surely as $n \rightarrow \infty$ [27, 6] for some bounded μ_k . It follows that $\|\mathbb{P}_0 - \mathbb{P}_\beta\|_{\text{TV}} \rightarrow 1$ for all $\beta > 2\mu_k$ [31]. Hence, the second moment bound is off by a k -dependent factor. For large k , $2\mu_k = \sqrt{2\log k} + O_k(1)$ and hence the factor is indeed bounded in k .

Behavior below the threshold. Let us stress an important qualitative difference between $k = 2$ and $k \geq 3$, for $\beta < \beta_k^{2\text{nd}}$. For $k \geq 3$, the two models are indistinguishable and any test is essentially as good as random guessing. Formally, for any measurable function $T : \otimes^k \mathbb{R}^n \rightarrow \{0, 1\}$, we have

$$\lim_{n \rightarrow \infty} [\mathbb{P}_0(T(\mathbf{X}) = 1) + \mathbb{P}_\beta(T(\mathbf{X}) = 0)] = 1. \quad (30)$$

For $k = 2$, our result implies that, for $\beta < 1$, $\|\mathbb{P}_0 - \mathbb{P}_\beta\|_{\text{TV}}$ is bounded away from 1. On the other hand, it is easy to see that it is bounded away from 0 as well, i.e.

$$0 < \liminf_{n \rightarrow \infty} \|\mathbb{P}_0 - \mathbb{P}_\beta\|_{\text{TV}} \leq \limsup_{n \rightarrow \infty} \|\mathbb{P}_0 - \mathbb{P}_\beta\|_{\text{TV}} < 1. \quad (31)$$

Indeed, consider for instance the statistics $S = \text{Tr}(\mathbf{X})$. Under \mathbb{P}_0 , $S \sim \mathcal{N}(0, 2)$, while under \mathbb{P}_β , $S \sim \mathcal{N}(\beta, 2)$. Hence

$$\liminf_{n \rightarrow \infty} \|\mathbb{P}_0 - \mathbb{P}_\beta\|_{\text{TV}} \geq \|\mathcal{N}(0, 1) - \mathcal{N}(\beta/\sqrt{2}, 1)\|_{\text{TV}} = 1 - 2\Phi\left(-\frac{\beta}{2\sqrt{2}}\right) > 0 \quad (32)$$

(Here $\Phi(x) = \int_{-\infty}^x e^{-z^2/2} dz / \sqrt{2\pi}$ is the Gaussian distribution function.) The same phenomenon for rectangular matrices ($k = 2$) is discussed in detail in [32].

5 Asymmetric tensor model

As before, we denote by $\mathbf{G} \in \otimes^k \mathbb{R}^n$ a tensor with independent and identically distributed entries $\mathbf{G}_{i_1, \dots, i_k} \sim \mathcal{N}(0, 1)$. We define the *asymmetric standard normal* noise tensor $\mathbf{Z} \in \otimes^k \mathbb{R}^n$ by

$$\mathbf{Z} = \sqrt{\frac{1}{n}} \mathbf{G}. \quad (33)$$

In particular, all entries are i.i.d. $\mathbf{Z}_{i_1, \dots, i_k} \sim \mathcal{N}(0, 1/n)$. With this normalization, we have of course

$$\mathbb{E}\{e^{\langle \mathbf{A}, \mathbf{Z} \rangle}\} = \exp\left\{\frac{1}{2n} \|\mathbf{A}\|_F^2\right\}. \quad (34)$$

Given $\lambda \in \mathbb{R}_{\geq 0}$, we consider observations $\mathbf{X} \in \otimes^k \mathbb{R}^n$ given by:

$$\mathbf{X} \equiv \lambda \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \dots \otimes \mathbf{v}_k + \mathbf{Z}, \quad (35)$$

with \mathbf{Z} a an asymmetric standard normal tensor, and $\mathbf{v}_1, \dots, \mathbf{v}_k$ independent and uniformly distributed over the unit sphere \mathbb{S}^{n-1} . In the case $k = 2$ we recover, again, the classical spiked model. Note that a further layer of generalization would be obtained by considering a ‘rectangular’ tensor $\mathbf{X} \in \mathbb{R}^{n_1} \otimes \dots \otimes \mathbb{R}^{n_k}$. We prefer to assume $n_1 = \dots = n_k$ to limit inessential technical complications.

We denote by $\mathbb{P}_\lambda = \mathbb{P}_\lambda^{(k)}$ the law of \mathbf{X} under the model (35)

Theorem 6. For $k \geq 2$, let $\lambda_k^{2\text{nd}} \equiv (k/2)^{1/2} \beta_k^{2\text{nd}}$, i.e.

$$\lambda_k^{2\text{nd}} \equiv \inf_{q \in (0, 1)} \sqrt{-\frac{k}{2q^k} \log(1 - q^2)}. \quad (36)$$

Assume $\lambda < \lambda_k^{2\text{nd}}$. Then, for any $k \geq 3$, we have

$$\lim_{n \rightarrow \infty} \|\mathbb{P}_\lambda - \mathbb{P}_0\|_{\text{TV}} = 0. \quad (37)$$

Further, for $\lambda < \lambda_2^{2\text{nd}} = 1$, \mathbb{P}_λ is contiguous with respect to \mathbb{P}_0 .

Proof. The proof is very similar to the one of Theorem 3. We will therefore limit ourselves to outline the main steps, and the differences with respect to the symmetric case. First we compute the Radon-Nikodym derivative of \mathbb{P}_λ with respect to \mathbb{P}_0 :

$$\Lambda = \int \exp \left\{ -\frac{n\lambda^2}{2} + n\lambda \langle \mathbf{X}, \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k \rangle \right\} \mu_n^{\otimes k}(\mathrm{d}\mathbf{v}). \quad (38)$$

Here we introduced the notation $\mu_n^{\otimes k}(\mathrm{d}\mathbf{v}) = \mu_n(\mathrm{d}\mathbf{v}_1) \cdots \mu_n(\mathrm{d}\mathbf{v}_k)$. Proceeding as in the proof of Eq. (22), we get

$$\mathbb{E}_0 \Lambda^2 = \int \exp \left\{ n\lambda^2 \prod_{i=1}^k \langle \mathbf{v}_i, \mathbf{e}_1 \rangle^k \right\} \mu_n^{\otimes k}(\mathrm{d}\mathbf{v}). \quad (39)$$

Now, let $G_\lambda : [-1, 1]^k \rightarrow \overline{\mathbb{R}}$ be defined by

$$G_\lambda(q_1, q_2, \dots, q_k) = \lambda^2 \prod_{i=1}^k q_i + \frac{1}{2} \sum_{i=1}^k \log(1 - q_i^2). \quad (40)$$

Invoking again Lemma 5, we obtain that for any open set $J \in [-1, 1]^k$

$$\int \exp \left\{ n\lambda^2 \prod_{i=1}^k \langle \mathbf{v}_i, \mathbf{e}_1 \rangle^k \right\} \mathbb{I} \left((\langle \mathbf{v}_1, \mathbf{e}_1 \rangle, \dots, \langle \mathbf{v}_k, \mathbf{e}_1 \rangle) \in J \right) \mu_n^{\otimes k}(\mathrm{d}\mathbf{v}) = \exp \left\{ n \max_{\mathbf{q} \in J} G_\lambda(\mathbf{q}) + o(n) \right\}. \quad (41)$$

The key observation is that, for $\lambda < \lambda_k^{2\text{nd}}$, we have $\max_{\mathbf{q} \in [-1, 1]^k} G_\lambda(\mathbf{q}) = 0$, with the maximum being uniquely achieved at $\mathbf{q} = 0$. Once this claim is proved, we can restrict the integral (39) to a neighborhood of $(\langle \mathbf{v}_1, \mathbf{e}_1 \rangle, \dots, \langle \mathbf{v}_k, \mathbf{e}_1 \rangle) = 0$ and obtain by an argument completely analogous to the symmetric case

$$k \geq 3 \Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}_0 \Lambda^2 = 1. \quad k = 2 \Rightarrow \limsup_{n \rightarrow \infty} \mathbb{E}_0 \Lambda^2 \leq \frac{1}{\sqrt{1 - \lambda^4}}.$$

To prove the above claim, and hence complete the proof, note that: $\mathbf{q} = 0$ is a local maximum of $G_\lambda(\mathbf{q})$; $G_\lambda(|q_1|, \dots, |q_k|) \geq G_\lambda(q_1, \dots, q_k)$; $G_\lambda(\mathbf{q}) \rightarrow -\infty$ if $\|\mathbf{q}\|_\infty \rightarrow 1$. It is therefore sufficient to prove that any other local maximum $\mathbf{q}_* \in [0, 1]^k \setminus \{0\}$ has $G_\lambda(\mathbf{q}_*) < 0$. The stationarity condition of G_λ reads

$$\lambda^2 \prod_{j \in [k] \setminus i} q_j = \frac{q_i}{1 - q_i^2}, \quad \forall i \in [k], \quad (42)$$

whence (after multiplying by q_i)

$$\frac{q_1^2}{1 - q_1^2} = \frac{q_2^2}{1 - q_2^2} = \cdots = \frac{q_k^2}{1 - q_k^2}. \quad (43)$$

since $x \mapsto x^2/(1 - x^2)$ is strictly monotone increasing on $(0, 1]$, we deduce that $q_1 = q_2 = \cdots = q_k$. It is therefore sufficient to check $G_\lambda(q, q, \dots, q) < 0$ for all $q \in (0, 1)$. This is guaranteed by $\lambda < \lambda_k^{2\text{nd}}$. \square

6 Related work

Detection problems with similar flavor to the Gaussian hidden clique problem have been studied over the years in several fields including computer science, physics and statistics. Typically, in such problems there is “planted” object with special properties along with random noise which makes the detection of the planted object a nontrivial task. In the classical $G(n, 1/2)$ planted clique problem, the computational problem is to find the planted clique (of cardinality k) efficiently (e.g., in polynomial time) where we assume the location of the planted clique is hidden and is not part of the input. There are several algorithms that recover the planted clique in polynomial time when $k = C\sqrt{n}$ where $C > 0$ is a constant independent of n [2, 12, 15, 16]. In [11] it is proven that a planted clique can be recovered in time $O(n^2 \log(n))$, whenever $C = e^{-1/2} + \epsilon$. The work of [2] demonstrates that it is possible to find a planted clique of size $c\sqrt{n}$ in time $n^{O(\log(1/c))}$. Despite significant effort, no polynomial time algorithm for this problem is known when $k = o(\sqrt{n})$. In the decision version of the planted clique problem, one seeks an efficient algorithm that distinguishes between a random graph distributed as $G(n, 1/2)$ or a random graph containing a planted clique of size $k \geq (2 + \delta) \log n$ (for $\delta > 0$; the natural threshold for the problem is the size of the largest clique in a random sample of $G(n, 1/2)$, which is asymptotic to $2 \log n$ [20]). No polynomial time algorithm is known for this decision problem if $k = o(\sqrt{n})$. There are several hardness results for computational problems in game theory [30] (see also [21]) and statistics [9] which are based on the alleged hardness of the the problem of distinguishing between a random graph distributed as $G(n, 1/2)$ to a random graph with a planted clique of size $k(n)$ (with $(2 + \delta) \log n < k(n) \ll \sqrt{n}$).

As another example, consider the following setting introduced by [5] (see also [1]): one is given a realization of a n -dimensional Gaussian vector $\mathbf{x} := (\mathbf{x}_1, \dots, \mathbf{x}_n)$ with i.i.d. entries. The goal is to distinguish between the following two hypotheses. Under the first hypothesis, all entries in \mathbf{x} are i.i.d. standard normals. Under the second hypothesis, one is given a family of subsets $C := \{S_1, \dots, S_m\}$ such that for every $1 \leq k \leq m$, $S_k \subseteq \{1, \dots, n\}$ and there exists an $i \in \{1, \dots, m\}$ such that, for any $\alpha \in S_i$, \mathbf{x}_α is a Gaussian random variable with mean $\mu > 0$ and unit variance whereas for every $\alpha \notin S_i$, \mathbf{x}_α is standard normal. (The second hypothesis does not specify the index i , only its existence). The main question is how large μ must be such that one can reliably distinguish between these two hypotheses. In [5], one considers two situations. In the first, α are vertices in a two dimensional grid of side length n and the family C is the set of all directed (i.e., with north or east steps only) paths of length l , starting from the bottom-left corner. In the second situation treated in [5], the α s correspond to the vertices of a binary tree, and again the family C consists of loopless paths starting at the root. In [5], both the min-max and Bayesian (with uniform choice of i in C) setups are considered. Other choices of C are considered in [1]: the family of all subsets of size k , the set of all perfect matching in a given graph and other examples. These detection problems have practical applications-see [5] for details.

The Gaussian hidden clique problem is related to various applications in statistics and computational biology [7, 26]. That detection is statistically possible when $L \gg \log n$ was established in [1] (the authors consider the case where all diagonal elements are zero, but since the detection algorithm can simply ignore the diagonal elements, their results apply to our setting as well). Similar results for the asymmetric case were obtained by [28]. In terms of *polynomial time* detection, [11] show that detection is possible when $L = \Theta(\sqrt{n})$ for the symmetric cases. As noted, no polynomial time algorithm is known for the Gaussian hidden clique problem when $k = o(\sqrt{n})$. In [1] it was hypothesized that the Gaussian hidden clique problem should be difficult when $L \ll \sqrt{n}$. More specifically [1, Pg. 16, second paragraph] comment that “it seems likely that designing an efficient

test in the normal setting will prove as difficult as it proved for planted cliques". Supporting evidence for this assertion was provided in [28], who proved that distinguishing between a planted model similar to the Gaussian planted clique model studied in our work and the random case (with entries being independent standard Gaussians) is at least as hard as distinguishing between a graph containing a planted clique and a graph distributed the random graph $G(n, 1/2)$.

There is a large body of work in RMT that addresses the effect of low rank perturbations on various properties the spectrum and eigenvectors of Wigner matrices (e.g., [17, 18, 25]), mostly in studying the almost sure limits and limit distributions of extremal eigenvalues and eigenvectors.

The closest results to ours are the ones of [32]. In the language of the present paper, these authors consider a rectangular matrix of the form $\mathbf{X} = \lambda \mathbf{v}_1 \mathbf{v}_2^\top + \mathbf{Z} \in \mathbb{R}^{n_1 \times n_2}$ whereby \mathbf{Z} has i.i.d. entries $\mathbf{Z}_{ij} \sim \mathcal{N}(0, 1/n_1)$, \mathbf{v}_1 is deterministic of unit norm, and \mathbf{v}_2 has entries which are i.i.d. $\mathcal{N}(0, 1/n_1)$, independent of \mathbf{Z} . They consider the problem of testing this distribution against $\lambda = 0$. Setting $c = \lim_{n \rightarrow \infty} \frac{n_1}{n_2}$, it is proved in [32] that the distribution of the singular values of \mathbf{X} under the null and the alternative are mutually contiguous if $\beta < \sqrt{c}$ and not mutually contiguous if $\beta > \sqrt{c}$. This (almost) correspond to the asymmetric model of Section 5, for the matrix case $k = 2$. While [32] derive some more refined results, their proofs rely on advanced tools from random matrix theory [19], while our proof is simpler, and generalizable to other settings (e.g. tensors).

7 Conclusion

In this work we considered detection problems for GOE matrices perturbed by a deterministic rank 1 matrix, including the Gaussian hidden clique problem. We have established that spectral methods stop being effective when the norm of the perturbation drops below a threshold, which translates in the Gaussian hidden clique problem to the size of the planted submatrix being smaller than $(1 - \epsilon)\sqrt{n}$. In identifying this threshold we have also addressed detectability issues in rank-one perturbations of matrices and tensors which might be of independent interest.

There are several open problems that arise from the current work. First, in the context of the Gaussian hidden clique problem, it would be interesting to provide an efficient algorithm for finding a planted submatrix when $L = o(\sqrt{n})$ or rule out certain algorithmic (non spectral) approaches for this problem. One direction is to study optimization methods such as semidefinite programming and various types of hierarchies (e.g., Lasserre, Sherali-Adams) in dealing with the hidden clique problem for $L = o(\sqrt{n})$.

A natural question is whether one can use the spectrum in order to distinguish between a graph distributed as $G(n, 1/2)$ and a random graph with a planted clique of size $L < (1 - \epsilon)\sqrt{n}$. Proving that one can or cannot distinguish between these cases using eigenvalues is a challenging open problem.

Finally, it might be interesting to study the limitations of spectral techniques for other problems such as coloring [3] and satisfiability [14].

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